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The binomial coefficient $\binom{n}{i} \equiv \frac{n!}{i!(n-i)!}$ for n and i non-negative integers with $i \le n$

The binomial coefficient gives a convenient way to represent the coefficients of the terms in $(x+y)^n$ with n a non-negative integer. Specifically, $(x+y)^n \equiv \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$

Examples follow here for n = 0, 1, 2, 3, 4, 5.

$$(x+y)^0 \equiv \binom{0}{0} x^0 y^0 \equiv \frac{0!}{0!0!} x^0 y^0 \equiv 1$$

$$(x+y)^1 \equiv \binom{1}{0} x^{1-0} y^0 + \binom{1}{1} x^{1-1} y^1 \equiv \frac{1!}{0!1!} x + \frac{1!}{1!0!} y \equiv x + y$$

$$(x+y)^2 \equiv \binom{2}{0} x^{2-0} y^0 + \binom{2}{1} x^{2-1} y^1 + \binom{2}{2} x^{2-2} y^2 \equiv \frac{2!}{0!2!} x^2 + \frac{2!}{1!1!} xy + y^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 \equiv \binom{3}{0} x^{3-0} y^0 + \binom{3}{1} x^{3-1} y^1 + \binom{3}{2} x^{3-2} y^2 + \binom{3}{3} x^{3-3} y^3$$

$$\equiv \frac{3!}{0!3!} x^3 + \frac{3!}{1!2!} x^2 y^1 + \frac{3!}{2!1!} x^1 y^2 + \frac{3!}{3!0!} y^3$$

$$\equiv x^3 + 3x^2 y + 3x y^2 + y^3$$

$$(x+y)^4 \equiv \binom{4}{0} x^{4-0} y^0 + \binom{4}{1} x^{4-1} y^1 + \binom{4}{2} x^{4-2} y^2 + \binom{4}{3} x^{4-3} y^3 + \binom{4}{4} x^{4-4} y^4$$

$$\equiv \frac{4!}{0!4!} x^4 + \frac{4!}{1!3!} x^3 y^1 + \frac{4!}{2!2!} x^2 y^2 + \frac{4!}{3!1!} x^1 y^3 + \frac{4!}{4!0!} y^4$$

$$\equiv x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4$$

$$(x+y)^5$$

$$\equiv \binom{5}{0} x^{5-0} y^0 + \binom{5}{1} x^{5-1} y^1 + \binom{5}{2} x^{5-2} y^2 + \binom{5}{3} x^{5-3} y^3 + \binom{5}{4} x^{5-4} y^4 + \binom{5}{5} x^{5-5} y^5$$

$$\equiv \frac{5!}{0!5!} x^5 + \frac{5!}{1!4!} x^4 y^1 + \frac{5!}{2!3!} x^3 y^2 + \frac{5!}{3!2!} x^2 y^3 + \frac{5!}{4!1!} x y^4 + \frac{5!}{5!0!} y^5$$

$$\equiv x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5x y^4 + y^5$$

The preceding expansions make apparent that the binomial coefficients for n = 0, 1, 2, 3, … can be elements of a $(n+1)\times(2n+1)$ matrix that for n = 5, for example, is the 6×11 matrix

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Conventionally, *Pascal's Triangle* refers to the triangular projection of the coefficients in that arrangement. The pattern that governs augmenting the matrix—i.e., extending Pascal's Triangle—is trivially apparent: each added row adds two columns; the first and last columns receive a 1 and, beginning with the matrix for n=2, alternating interior columns receive the sum of the two surrounding columns one row above.